

# On quantum symmetries of the non-ADE graph $F_4$

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## Abstract

We describe quantum symmetries associated with the  $F_4$  Dynkin diagram. Our study stems from an analysis of the (Ocneanu) modular splitting equation applied to a partition function which is invariant under a particular congruence subgroup of the modular group.

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## 1 Introduction

Modular invariant partition functions of the affine  $SU(2)$  conformal field theory models have been classified long ago [1]; they follow an *ADE* classification. There are three infinite series called  $A_n$ ,  $D_{2n}$ ,  $D_{2n+1}$  and three exceptional cases called  $E_6$ ,  $E_7$  and  $E_8$ . The terminology was justified from the fact that exponents of the corresponding Lie groups appear in the expression of the corresponding partition functions but, originally, this labelling using Dynkin diagrams was only a name since the diagram itself was not an ingredient in the construction. Later, and following in particular the work of [17][16], the corresponding field theory models were built directly from the data associated with the diagrams themselves.

About ten years ago, the occurrence of *ADE* diagrams in the affine  $SU(2)$  classification was understood in a rather different way. One observation (already present in reference [17]) is that the vector space spanned by the vertices of a diagram  $A_n$  possesses an associative and commutative algebra structure encoded by the diagram itself : this so-called “graph algebra” has a unit  $\tau_0$  (the first vertex), one generator  $\tau_1$  (the next vertex) and multiplication of a vertex  $\tau_p$  by  $\tau_1$  is given by the sum of the neighbors of  $\tau_p$ . So,  $\tau_1 \tau_p = \tau_{p-1} + \tau_{p+1}$  when  $p < n - 1$  and  $\tau_1 \tau_{n-1} = \tau_{n-2}$ . The structure constants of this algebra (which can be understood as a quantum version of the algebra of  $SU(2)$  irreps at roots of unity) are positive integers. In some cases ( $A_n$ ,  $D_{2n}$ ,  $E_6$ ,  $E_8$ ), the vector space spanned by the vertices of a chosen Dynkin diagram of type *ADE* also enjoys self-fusion, i.e., admits, like  $A_n$  itself, an associative algebra structure, with structure constants that are positive integers, together with a multiplication table “related” to the graph algebra of the corresponding  $A_n$ . Another important observation is that this vector space is always a module over the graph

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algebra of  $A_n$  where  $n + 1$  is the Coxeter number of the chosen diagram. For instance the vector spaces spanned by vertices of the diagrams  $E_6$  and  $D_7$  are modules over the graph algebra of  $A_{11}$  (their common Coxeter number is 12). The Ocneanu construction [12] associates with every *ADE* Dynkin diagram  $G$  a special kind of weak Hopf algebra (or quantum groupoid). This bialgebra  $B(G)$  is finite dimensional and semi-simple for its two associative structures. Existence of a coproduct on the underlying vector space (and on its dual) allows one to take tensor product of irreducible representations (or co-representations) and decompose them into irreducible components. One obtains in this way two – usually distinct – algebras of characters. The first, called the fusion algebra of  $G$ , and denoted  $A(G)$ , can be identified with the graph algebra of  $A_n$ , where  $n + 1$  is the Coxeter number of  $G$  (this number is defined in a purely combinatorial way, and does not require any reference to the theory of Lie algebras or Coxeter groups). The second algebra of characters is called the algebra of quantum symmetries and denoted  $Oc(G)$ ; it is an associative – but not necessarily commutative – algebra with two (usually distinct) generators. It comes with a particular basis, and the multiplication of its basis elements by the two generators is encoded by a graph called the Ocneanu graph of  $G$ . The quantum groupoid associated with  $G$  is “special” in several ways. For instance, each one of the two algebras  $A(G)$  and  $Oc(G)$  is a bimodule over the other; this bimodule structure is encoded by a set of “toric matrices” and one of them can be identified with the modular invariant partition function for a physical system. The others, which are not modular invariant, can be physically interpreted, in the framework of Boundary Conformal Field Theory, as partition functions in the presence of defects, see [18]. Physical applications of this general formalism may be found in statistical mechanics [15], string theory or quantum gravity, but this is not the subject of the present paper.

Modular invariance can be investigated either directly, by associating vertices  $\tau_p$  of the graph  $A_n$  with explicit functions  $\chi_p$  defined on the upper - half plane (the characters of an affine Kac-Moody algebra), or more simply, by checking commutation relations between a particular toric matrix and the two generators  $S$  and  $T$  of the  $SL(2, \mathbb{Z})$  group in a particular representation (Verlinde - Hurwitz, [11] [21] ). The matrix  $S$ , as given by the Verlinde formula, is actually a non-commutative analog of the table of characters for a finite group and can be obtained directly from the multiplication table of the graph algebra of  $A_n$ .

The theory that was briefly summarized above can be generalized from  $SU(2)$  to  $SU(N)$  or, more generally, to any affine algebra associated with a chosen Lie group. The familiar simply laced *ADE* Dynkin diagrams are associated with the affine  $SU(2)$  theory, but more general Coxeter-Dynkin systems (each system being a collection of diagrams together with their corresponding quantum groupoids) can be obtained [13] . For instance the Di Francesco - Zuber diagrams [8] [9] are associated with the  $SU(3)$  system.

What notion comes first ? The chosen diagram, member of some Coxeter-Dynkin system ? Its corresponding quantum groupoid ? Or the associated modular invariant ? The starting point may be a matter of taste... However, from a practical point of view, it is probably better to start from the combinatorial data provided by a given modular invariant. Indeed, apart from  $SU(2)$  and, to some extent,  $SU(3)$ , the Coxeter-Dynkin system itself is not a priori known, whereas existence of several algorithms, mostly due to T. Gannon, allows one to explore up to rather high levels the possibly new modular invariants associated with every choice of an affine Kac - Moody algebra. The primary data is then a given modular invariant – a sesquilinear form in the characters of some affine Kac-Moody algebra. From the resolution, over the positive integers, of a particular equation, called “equation of modular splitting” (more about it later), one can determine first the set of toric matrices, and then the algebra of quantum symmetries. The associated Dynkin diagram – an *ADE* diagram in the case of the  $SU(2)$  system – or, more generally, the particular member of some higher Coxeter-Dynkin system becomes an *outcome* of the construction, not a starting point. The  $SU(3)$  system of diagrams was already essentially determined by [8], [9], and recovered by A. Ocneanu, using this modular splitting technique, together with the known classification of  $SU(3)$  modular invariants obtained by T.Gannon [10]. It was also the route followed by [14] in his classification of the  $SU(4)$  system. In all these examples, starting from a modular invariant partition function,

one obtains a diagram that is simply laced (i.e., an *ADE* diagram, for the  $SU(2)$  system), or is a generalization of what could be defined as a “simply laced diagram”, for the higher systems.

In the present paper we are only interested in the affine  $SU(2)$  system and we want to start from a partition function which is not modular invariant but which is nevertheless invariant under some particular congruence subgroup. Our starting point is the so-called “ $F_4$  partition function” which appears as a kind of  $\mathbb{Z}_2$  orbifold of the  $E_6$  modular invariant (its name comes from the fact that exponents of the Lie group  $F_4$  appear in its expression). After the work of [7], this partition function was discussed in [22]. Our purpose is neither to propose any physical conformal field theory model that would lead to this expression nor to investigate its analytical properties but rather to analyze the equation of modular splitting corresponding to this particular choice of a modular non-invariant expression and see what algebraic structure it gives. Starting from this data, we shall determine, in turn, a set of toric matrices and an algebra of quantum symmetries (described by an Ocneanu graph). Actually we do not even suppose that we “know” what the diagram  $F_4$  is : it will appear as a subgraph of the graph of quantum symmetries. We shall not try here to generalize our study to cover other non-ADE cases of the  $SU(2)$  system, and shall not investigate, either, the non simply laced diagrams belonging to higher systems. Since we want to focus on the modular splitting technique itself, we shall not try to use or modify, in this non-simply laced case, the definitions of the product and coproduct that usually lead to a quantum groupoid structure, but the fact that one can solve the equation of modular splitting, define toric matrices and determine an algebra of quantum symmetries together with two algebras of characters obeying the usual quadratic sum rules is by itself a non trivial result that suggests further developments.

The algorithms that we use in order to solve the relevant equations and determine the algebraic quantities of interest have been developed by the authors, but it may well be that more efficient techniques have been found by others<sup>1</sup>. If so, this information is not available. Although devoted to the study of a particular member of an unusual class (a non-*ADE* example), we believe that the present paper may be of interest for the reader who wants to see how the general technique based on the equation of modular splitting works, since it does not seem to be documented in the literature. Apart from considerations of computing efficiency, it should not be too hard to adapt the following analysis to study other cases, simply laced or not, associated with any Coxeter-Dynkin system<sup>2</sup>.

Before ending this introduction, let us stress the fact that the present paper does not require any knowledge of the theory of Lie algebras or quantum groups (or quantum groupoids). All the algebras used in the following are associative algebras, not Lie. A sentence like “the algebra  $A_n$ ” means actually “the associative graph algebra corresponding to the diagram  $A_n$ ” and we usually identify a given Dynkin diagram with the vector space or the  $\mathbb{Z}$ -module spanned by its vertices.

## 2 Toric matrices from modular splitting

### Invariance of the partition function

For the  $SU(2)$  system, there are three modular invariant partition functions at Coxeter number  $\kappa = 12$ : they are respectively associated with the diagrams  $A_{11}$ ,  $D_7$  and  $E_6$ . The first case (also called “diagonal”) is given by

$$Z_{A_{11}} = \sum_{n=0}^{10} |\chi_n|^2$$

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<sup>1</sup>The results presented long ago in [12] or [14], for instance, require the use of analogous techniques.

<sup>2</sup>See the forthcoming paper [20] where an exceptional simply laced example of the  $SU(4)$  system is studied.

Modular invariance of this expression can be explicitly checked by performing the transformations  $T : \tau \rightarrow \tau + 1$  and  $S : \tau \rightarrow -1/\tau$ ; to do that, one can use an explicit expressions for the eleven characters  $\chi_p$  of affine  $SU(2)$  at level 10, for instance in terms of theta functions. Another possibility, which is simpler, is to represent  $S$  and  $T$  by  $11 \times 11$  matrices, namely  $S_{ij} = \sqrt{\frac{2}{\pi}} \sin(\pi \frac{(i+1)(j+1)}{\kappa})$ ,  $0 \leq i, j \leq \kappa - 2$ , and  $T_{ij} = \exp[2i\pi(\frac{(j+1)^2}{4\kappa} - \frac{1}{8})] \delta_{ij}$ , and check that they commute with the modular matrix  $M$  associated with  $Z$  (the relation between the two being of course,  $Z = \bar{\chi} M \chi$ ). Commutation is obvious since  $M$  is the diagonal unit matrix  $\mathbb{1}_{11}$ .

The  $E_6$  partition function is given by

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2$$

Its modular invariance can be checked as above but the associated modular matrix is non-trivial.  $Z_{E_6}$  is a sum of three modulus squared “generalized characters”  $\lambda_1 = \chi_0 + \chi_6$ ,  $\lambda_2 = \chi_3 + \chi_7$ , and  $\lambda_3 = \chi_4 + \chi_{10}$ .  $M$  has 12 non - zero entries equal to 1. This is related to the fact that the Ocneanu graph has 12 points, three of them being ambichiral, the corresponding algebra being commutative.

We now turn to  $F_4$ , which is our object of study. The partition function is

$$Z_{F_4} = |\chi_0 + \chi_6|^2 + |\chi_4 + \chi_{10}|^2$$

The fact that it is *not* modular invariant is obvious from the modular transformations of the generalized characters of  $E_6$ : Under  $\tau \rightarrow -1/\tau$ ,  $\lambda_1 \rightarrow (1/2)(\lambda_1 + \lambda_2) - (1/\sqrt{2})\lambda_2$ ,  $\lambda_2 \rightarrow (1/\sqrt{2})(\lambda_3 - \lambda_1)$ ,  $\lambda_3 \rightarrow (1/2)(\lambda_1 + \lambda_2) + (1/\sqrt{2})\lambda_2$ . and under  $\tau \rightarrow \tau + 1$ ,  $\lambda_1 \rightarrow e^{\frac{19i\pi}{24}} \lambda_1$ ,  $\lambda_2 \rightarrow e^{\frac{10i\pi}{24}} \lambda_2$ ,  $\lambda_3 \rightarrow e^{\frac{-5i\pi}{24}} \lambda_3$ . However, these relations also show that it is invariant under the transformations  $\tau \rightarrow \tau + 2$  and  $\tau \rightarrow \frac{\tau}{2\tau+1}$  that span a congruence subgroup<sup>3</sup>  $\Gamma_0^{(2)}$  of  $SL(2, \mathbb{Z})$  at level 2. It is actually easier to show this by checking explicitly that the modular matrix associated with  $Z_{F_4}$ , namely

$$M = \begin{pmatrix} 1 & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & 1 \end{pmatrix}$$

commutes with the generators  $T^2$  and  $ST^{-2}S$  of  $\Gamma_0^{(2)}$ . Notice that the exponents of the Lie group  $F_4$  appear on the diagonal of  $M$ , hence the name chosen for this partition function. From the fact that  $Z_{F_4}$  is a sum of two squares, one expects two ambichiral points in the Ocneanu graph (indeed it will turn out to be so).  $M$  has 8 non - zero entries equal to 1, this could suggest that the corresponding Ocneanu graph has the same number of vertices... however, as we shall see, this is *not* so.

## Toric matrices (definition)

Let  $m, n, \dots$  (rather than  $\tau_m, \tau_n$ , or  $\chi_m, \chi_n$ ) denote the vertices of  $A_{11}$  and  $x, y, \dots$  the vertices of the Ocneanu graph  $Oc$  to be found. As already mentioned,  $Oc$  should be a bi-module over  $A_{11}$  so

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<sup>3</sup>We do not use the corresponding projective group

that there exist a collection of  $11 \times 11$  toric matrices  $W_{x,y}$ , with positive integer entries, such that

$$m x n = \sum_y (W_{x,y})_{m,n} y$$

These are “toric matrices with two twists” ( $x$  and  $y$ ). If  $o$  is the number of vertices in  $Oc$  – it will be determined later – the number of such matrices is of course  $o^2$  but many of them may coincide. Since  $A_{11}$  and  $Oc$  play a dual role, it is useful to introduce the  $11^2$  matrices  $V_{m,n}$  of size  $o \times o$  by

$$(V_{m,n})_{x,y} = (W_{x,y})_{m,n}$$

The algebra of quantum symmetries has a unit, called  $\underline{0}$ , a particular vertex of  $Oc$ , so that we have also “matrices with one twist”  $W_{0,y}$  and  $W_{x,0}$ . When the example under study corresponds to a simply laced situation (for instance the  $ADE$  cases of the usual  $SU(2)$  system) and if the algebra  $Oc$  is commutative, one shows that  $W_{x,y} = W_{y,x}$ . However, we are now in a new situation and should keep our mind open. The resolution of the general modular splitting equation will, in any case, determine all these quantities.

## Equations of modular splitting

The general equation of modular splitting expresses associativity of a bi-module structure and reads

$$(m n) x (p q) = (m (n x p) q)$$

The products  $(m n)$  or  $(p q)$  belong to the fusion algebra, i.e., for instance, to  $A_{11}$ , and involve its structure constants defined by  $m n = \sum_p N_{m,n}^p p$ . They are completely symmetric in their three indices. These positive integers are determined, recursively, by the – truncated –  $SU(2)$  algebra of compositions of spins. For  $A_{11}$  one obtains 11 matrices  $N_m$  of size  $11 \times 11$ , determined by the equations  $N_1 N_m = N_{m-1} + N_{m+1}$ , with  $N_0 = \mathbb{1}_{11}$  and  $N_1$ , the adjacency matrix of the diagram  $A_{11}$ . One obtains in particular,  $N_1 N_{10} = N_9$  since  $N_{11}$  vanishes. Of course  $N_{m,n}^p = (N_m)_n^p$ .

Using toric matrices, the general modular splitting equation reads therefore

$$\sum_{m'',n''} (N_{m''})_{m',m} (N_{n''})_{n',n} (W_{x,y})_{m''n''} = \sum_z (W_{x,z})_{mn} (W_{z,y})_{m'n'}$$

This general equation, valid for any simply laced graph belonging to a Coxeter-Dynkin system, together with its proof and interpretation, in terms of associativity of a bi-module structure, was obtained in the thesis [19]; it generalizes the “modular splitting equation” described in [14]. The later can be obtained from the former by setting  $x = y = 0$ , so that its left hand side involves only known quantities, namely the matrix  $M = W_{0,0}$  associated with the given graph (in the simply laced cases, it is *the* modular invariant), and the fusion coefficients. It reads

$$\sum_{m'',n''} (N_{m''})_{m',m} (N_{n''})_{n',n} M_{m''n''} = \sum_z (W_{z,0})_{m'n'} (W_{0,z})_{mn}$$

or, in terms of tensor products,

$$\sum_{mn} N_m \otimes N_n M_{m,n} = \sum_{z \in Oc} \tau \circ (W_{z,0} \otimes W_{0,z})$$

Here  $\tau$  denotes a tensorial flip (compare explicit indices in the two previous equations). The left and side – call it  $K$  – is therefore a known matrix of size  $121 \times 121$  (in this case), with positive

integer entries, and the right hand side involves a set of toric matrices (with one twist), to be determined. Each term of this right hand side should be a matrix of rank 1 with positive integer coefficients. Only one member of this family is a priori known, namely  $M = W_{0,0}$  which is our initial data. The indexing set on the right hand side of the modular splitting equation defines the set of vertices of the Ocneanu graph. The problem at hand is analogous to those related with convex decompositions in abelian monoids. In the simply laced cases (where  $W_{x,0} = W_{0,x}$  for all  $x$ ), each term appearing on the right hand side is a tensor square composed with the flip. In the non simply laced case, as we shall see, the situation is slightly more complicated. Notice that, in any case, calling “toric vectors”  $w_{x,y}$  the line-vectors obtained by “flattening” the toric matrices  $W_{x,y}$  (i.e.,  $(w_{x,y})_k = W_{x,y}[p, q]$  with  $k = (p-1)11 + q$ ), one can write the modular splitting equation as follows:  $K = \sum_z K_z$  where each matrix  $K_z$  – of size  $121 \times 121$  with positive integral entries – should be of rank 1, and its  $k$ -th line is equal to

$$K_z[k] = (w_{z,0})_k w_{z,0}$$

## Resolution of the equations

The algorithm used to solve this set of equations (over the positive integers) is slightly different for the known simply-laced case and for the example that we study in this paper. Let us first summarize what we would do in the usual situation and call  $n$  the total number of vertices of the corresponding fusion graph.

### The simply laced case

- The first line of  $K$  – a line vector with  $n^2$  components – is just the “flattened” invariant matrix  $M$ .
- One does not assume that  $W_{x,y}$  is equal to  $W_{y,x}$  but takes nevertheless  $W_x \doteq W_{x,0} = W_{0,x}$ .
- The rank  $r(K)$  of  $K$  can be calculated. This tells us that the dimension of the vector space spanned by the  $n^2$  lines of  $K$  can be expanded on a set of  $r(K)$  basis vectors  $w_x = w_{x,0}$  (the toric vectors). Once a toric vector  $w$  is found, the corresponding toric matrix is obtained by partitioning its entries into  $n$  lines of length  $n$ .
- The problem is to determine whether a given line of  $K$  is a toric vector or if it is a positive integral linear combination of such vectors, and in that later case, one has to find the number of such terms in the sum. We choose the (non canonical) scalar product for which the basis of toric vectors is orthonormal; for every line of  $K$  we write  $K[p] = \sum_x a(x) w_x$  and the norm square of the vector  $K[p]$  is therefore  $\sum a(x)^2$ . The fundamental observation is that this number (call it  $\ell[p]$ ) is equal to the diagonal matrix element  $K[p, p]$ .
- Since  $K$  is known, we first consider the list of values of  $p$  for which  $K[p, p] = 1$ . Since several line vectors  $K[p]$  and  $K[p']$  may coincide for distinct values of  $p, p'$ , we actually build a restricted list. For  $p$  in this list, every line vector  $K[p]$  is then automatically a toric vector.
- We next consider the list (actually a restricted list, as above) of values of  $p$  for which  $K[p, p] = 2$ . The corresponding line vectors  $K[p]$  should be the sum of two toric vectors, and there are three cases. Either  $K[p]$  is the sum of two already determined toric vectors, or it is the sum of an already determined toric vector and a new one, else it is equal to twice a new toric vector. For every value of  $p$  belonging to the new restricted list, it is enough to calculate the set of differences  $K[p] - w_x$  where  $w_x$  runs in the set of the already determined toric vectors, and impose that all the components of such differences should be positive integers.

- The next step is to consider the set of values of  $p$  for which  $K[p, p] = 3$ , etc. and generalize the previous discussion in an straightforward way. The process stops, ultimately, since the rank of the system is finite. At the end, we obtain a set of  $r(K)$  toric vectors which are either lines of  $K$  or linear combinations of lines of  $K$ .
- The integer  $r(K)$  may be strictly smaller than the number  $o$  of vertices of the Ocneanu graph. This happens when distinct quantum symmetries  $x$  are associated with the same toric matrix  $W_x$ . This is for example the case of the graph  $D_4$  where the rank is 5 but where  $o = 8$ . A method to determine such multiplicities is to plug the results for  $w_x$  (actually for the matrices  $W_x$ ) back into the modular splitting equation. If there are no multiplicities, this equation is readily checked. If it does not hold, one has to introduce appropriate multiplicities in the right hand side (introduce unknown coefficients and solve). In the case of  $E_6$ , the rank is 12, the final list of toric vectors is obtained from lines 1, 2, 3, 10, 11, 12, 13, 14, 21, 22, 23, 4 – 10 of  $K$ , the last one being equal to a difference of two lines<sup>4</sup>, and the modular splitting equation holds “on the nose”, so  $o = 12$  also. In the case of  $D_4$ , the rank is 5, the list of toric vectors is  $K[1], K[2], K[6], K[3]/2, K[7]/3$  but the modular splitting equation holds only by introducing multiplicities 2 and 3 for the last two<sup>5</sup>, the number of quantum symmetries  $o$  is therefore 8.

### The case of $F_4$ (non simply laced)

- As usual, the first line  $K[1]$  of  $K$  – a line vector with 121 components – is just the “flattened” invariant matrix  $M$ .
- The rank of  $K$  is 20. We present the results according to the decomposition number  $\ell[p]$  relative to the line vector  $K[p]$  of  $K$ , with  $p$  running from 1 to 121. The twenty toric vectors can be taken as follows.  $\ell[p] = 1$ :  $w[p] = K[p]$  with  $p = 1, 2, 3, 10, 11, 12, 13, 14, 21, 22, 23, 24$ .  
 $\ell[p] = 2$ :  $w[4] = K[4] - w[10] = K[4] - K[10]$ ,  $w[15] = K[15] - w[21] = K[15] - K[21]$ ,  $w[25] = K[25]/2$ ,  $w[34] = K[34] - w[22] = K[34] - K[22]$ ,  $w[35] = K[35] - w[21] = K[35] - K[21]$ .  
 $\ell[p] = 3$ :  $w[26] = (K[26] - w[24])/2 = (K[26] - K[24])/2$ ,  $w[36] = (K[36] - w[14])/2 = (K[36] - K[14])/2$ .  
 $\ell[p] = 5$ :  $w[37] = (K[37] - w[13] - w[15] - w[35])/2 = K[37] - K[13] - K[15] + 2K[21] - K[35])/2$
- We re - label the toric vectors in such a way that the index  $x$  of  $w_x$  runs from 0 to 19.

$$\begin{aligned}
w_0 &= w[1], w_1 = w[2], w_2 = w[3], w_3 = w[4], w_4 = w[10], \\
w_5 &= w[11], w_6 = w[12], w_7 = w[13], w_8 = w[14], w_9 = w[15], \\
w_{10} &= w[21], w_{11} = w[22], w_{12} = w[23], w_{13} = w[24], w_{14} = w[25], \\
w_{15} &= w[26], w_{16} = w[34], w_{17} = w[35], w_{18} = w[36], w_{19} = w[37]
\end{aligned}$$

There is a one to one correspondence between toric matrices  $W_{x,0}$  and the previously determined toric vectors  $w_x$  which have 121 components : just partition them into 11 lines of 11 elements. At a later stage we shall write the generators  $O_x$  of the algebra  $Oc$  in terms of tensor products, for this reason, the following distinct notations appear in this paper:

$W_x = W_{x,0} = W_{ab'}$  whenever  $O_x = a \otimes b$ . We hope that no confusion should arise between the notations  $W_{ab'}$  and  $W_{x,y}$ .

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<sup>4</sup>Using the notations of [2] or [3], these toric vectors correspond respectively to  $W_{00}, W_{01}, W_{02}, W_{05}, W_{40}, W_{10}, W_{11}, W_{21}, W_{51}, W_{50}, W_{20}, W_{30}$ .

<sup>5</sup>Using the notations of [12] or [3], they correspond respectively to  $W_0, W_{1\epsilon}, W_1, W_2 = W'_2, W_\epsilon = W_{2\epsilon} = W'_{2\epsilon}$





- As expected from the presence of the  $1/2$  coefficients in the list giving the vectors  $w[p]$ , the modular splitting equation does not hold if we impose  $W_{x,0} = W_{0,x}$  and sum only on the corresponding 20 terms on the right hand side. One possibility is to introduce a multiplicity two for entries  $w_{14} = w[25], w_{15} = w[26], w_{18} = w[36], w_{19} = w[37]$ ; this indeed works, in the sense that the modular splitting equation is then satisfied. With such a choice, the number of quantum symmetries would be  $o = 24$ , rather than 20. However, the algebra of quantum symmetries later determined by this choice incorporates several arbitrary coefficients that cannot be fixed by requirements of positivity and integrality alone. Since we are in “Terra Incognita” (namely quantum symmetries of non simply laced diagrams), we prefer to explore another possibility, which also allows us to solve the modular splitting equation and leads to a nice algebra of quantum symmetries (and, as we shall see, to the emergence of the diagram  $F_4$ ). Our choice is to keep only the previously determined 20 terms, no more, but without imposing equality of  $W_{x,0}$  and  $W_{0,x}$ . This choice is natural in view of the fact that these matrices actually “count” a number of essential paths between the origin 0 and  $x$  on the Ocneanu graph itself, and the fact that in the present case, the graph is not symmetric (all edges are not bi-oriented). In general, the solution to the modular splitting equation, for a given invariant matrix  $M$  is not necessarily unique, although some later considerations may impose extra conditions that, ultimately, lead to rejection of one or another solution. At the moment, we investigate one solution (which is both minimal in terms of number of quantum symmetries and natural from the path interpretation point of view) and explore the consequences.

In order for the modular splitting equation to be satisfied, we therefore take  $W_{0,x} = 2W_{x,0}$  when  $x = 14, 15, 18, 19$  and  $W_{0,x} = W_{x,0}$  for the others. As we shall see, this corresponds to the fact that the  $F_4$  diagram contains an oriented edge.

### 3 Quantum symmetries and Ocneanu graph

#### Determination of the two chiral generators $O_1$ and $O'_1$

Call  $K_0$  the rectangular matrix ( $121 \times 20$ ) obtained by decomposing each line vectors of  $K$  on the (flattened) toric matrices. For instance the fourth line  $K[4]$  of  $K$  is equal to  $w[4] + w[5]$ , so its components are  $(0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ .

Call  $L_0$  the rectangular matrix obtained by transposing the matrix ( $20 \times 121$ ) obtained by flattening each component of the column vector (twenty lines) containing the toric matrices.

When  $W_{x,0} = W_{0,x}$  it is easy to see that  $K_0 = L_0$  but this is not so in the present case.

If  $E_0$  denotes the essential matrix (also called “intertwiner”) associated with the origin of an  $ADE$  diagram (for the  $SU(2)$  system or higher generalizations), and if  $G_1$  denotes the corresponding adjacency matrix, it is so that  $E_1 \doteq N_1 \cdot E_0 = E_0 \cdot G_1$  where  $N_1$  is the generator of the fusion algebra (adjacency matrix of the appropriate  $A_n$  diagram).  $E_1$  coincides with the essential matrix associated with the next vertex (after the origin) and describes essential paths emanating from it.

We have the following analogy:  $K_0$  (or  $L_0$ ) play the same role as  $E_0$ , but now  $G_1$  should be replaced by one of the two generators of the algebra of quantum symmetries, and  $N_1$  should be replaced by  $N_0 \otimes N_1$  (so we replace the fusion algebra by its tensor square). In other words, we determine the generator  $O_1$  by solving the intertwining equation

$$N_0 \otimes N_1 \cdot K_0 = K_0 \cdot O_1$$

The other chiral generator  $O'_1$  is determined by solving the same equation, but replacing  $N_0 \otimes N_1$  by  $N_1 \otimes N_0$ . At this level it is interesting to recall that, in this analogy between the vector space of a diagram and its algebra of quantum symmetries, the fusion algebra should be replaced by its tensor square, and the role of fused adjacency matrices (the  $F_{ab}$  matrices of references [3] that represent the action of  $A_n$  on a given diagram) is played by the toric matrices themselves.

In the present situation (non simply laced), we could hesitate between  $L_0$  and  $K_0$ , but the choice actually does not matter : it turns out that this arbitrariness corresponds to the arbitrariness in the association between the asymmetric  $F_4$  graph and a particular adjacency matrix or its transpose.

In general, after having solved the modular splitting equation (and there is no necessarily uniqueness of the solution), we have to solve a generalized intertwining identity (the one just given) in order to find the two chiral generators. Notice that the solution could be non unique, even after imposing integrality and positivity, but in the present case the solution is unique and we list<sup>6</sup> below the two matrices  $O_1$  and  $O'_1$ , of dimension  $20 \times 20$ , which solve these equations.

$$O_1 = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}$$

$$O'_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Now that we have obtained the two algebraic generators, the algebra  $Oc$  that they span is determined as well (take linear combinations of powers and products). However we shall exhibit a particular basis of linear generators. To find them, one possibility is to first determine the set of matrices  $V_{m,n}$  that describe, in a dual way, the full set of toric matrices  $W_{x,y}$  with two twists.

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<sup>6</sup>The reader already recognizes, from the structure of  $O_1$ , two subdiagrams of type  $E_6$  and two others of type  $F_4$

## Determination of the toric matrices with two twists

The  $V_{m,n}$  are obtained by solving the general intertwining equation (notice that  $O_1 = V_{0,1}$  and  $O'_1 = V_{1,0}$ ).

$$N_m \otimes N_n \cdot K_0 = K_0 \cdot V_{m,n}$$

The solution is unique. Of course, we shall not list these  $11^2$  square matrices of dimension  $12 \times 12$  and we shall not give, either, the list of toric matrices with two twists, but remember that they are determined by the relation

$$(W_{x,y})_{m,n} = (V_{m,n})_{x,y}$$

One can then check that the generalized equation of modular splitting (the one that involves the  $W_{x,y}$  rather than the  $W_{x,0}$ ) is satisfied.

## Determination of the linear generators $O_x$ of $O_c$

The structure constants  $O_{xyz}$  are defined by the equations

$$W_{y,x} = \sum_z O_{xyz} W_z$$

where  $W_z = W_{z,0}$ . Notice that, in the present case,  $O_{xyz} = O_{zyx}$  but  $O_{xyz} \neq O_{yxz}$  in general. Matrices  $O_x$  are defined by their coefficients as follows :  $(O_x)_{yz} = O_{xyz}$ . We have

$$O_x O_y = \sum_z O_{xzy} O_z$$

and any two generators  $O_x$  and  $O_y$  commute, because of the symmetry properties of the structure constants.

One could be tempted to consider the (non-commutative) family of matrices  $Z_y$  defined by the equation  $(Z_y)_{xz} = O_{xyz}$  but one can see that this family is not multiplicatively closed; moreover  $Z_0$  does not even coincide with the identity matrix, since it has diagonal coefficients equal to 2 in positions  $x = 14, 15, 18, 19$ .

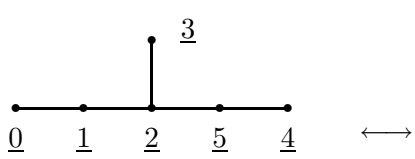
When the Ocneanu diagram possesses geometric symmetries, for instance in the case of the  $D$  diagrams, it may be that the general solution involve parameters that should be fixed by imposing positivity and integrality, and that one solution is only determined up to a discrete transformation reflecting the classical symmetries (this amounts to re-label the vertices  $x$ ). In the present case, however, everything is perfectly determined and we obtain the twenty generators of the Ocneanu diagram – They are  $20 \times 20$  matrices. Rather than giving this list explicitly (it would be typographically heavy !), we shall express them in terms of the already known and explicitly given chiral generators  $O_1$  and  $O'_1$ . Call  $O_0$  the unit matrix  $\mathbb{1}_{20}$ .

$O_0$ $O_1$ $O_2 = O_1^2 - O_0$ $O_5 = O_1^4 - 4O_1^2 + 2O_0$ $O_4 = O_1 \cdot O_5$ $O_3 = O_2 \cdot O_1 - O_4 - O_1$	$O_6 = O'_1$ $O_7 = O_1 \cdot O'_1$ $O_8 = O_2 \cdot O'_1$ $O_9 = O_3 \cdot O'_1$ $O_{10} = O_4 \cdot O'_1$ $O_{11} = O_5 \cdot O'_1$	$O_{12} = O_6 \cdot O'_1 - O_0$ $\downarrow$ $O_{13} = O_{12} \cdot O_1$ $O_{14} = O_{13} \cdot O_1 - O_{12}$ $O_{15} = O_{14} \cdot O_1 - 2O_{13}$ $O_{14} = O_{15} \cdot O_1 (check)$	$O_{16} = O_{12} \cdot O'_1 - O_6 - O_{11}$ $\downarrow$ $O_{17} = O_{16} \cdot O_1$ $O_{18} = O_{17} \cdot O_1 - O_{16}$ $O_{19} = O_{18} \cdot O_1 - 2O_{17}$ $O_{18} = O_{19} \cdot O_1 (check)$
--	--	--	--

The full multiplication table (that we don't display because it is  $20 \times 20$ ) defines a 20 dimensional algebra  $Oc$  with linear generators  $O_x$ , with  $x \in \{0, 1, 2, 3, \dots, 18, 19\}$ . It is generated, as an algebra, by the two matrices associated with vertices 1 and 6 (called *chiral generators*). We call it “algebra of quantum symmetries of  $F_4$ ”. Multiplication of any single linear generator  $O_x$  by the two chiral ones is encoded by a graph: the Ocneanu graph of  $F_4$ . It will be described later.

## The Ocneanu algebra as a the tensor square of a graph algebra

For all simply laced diagrams belonging to the  $SU(2)$  system or to an higher system, the algebra of quantum symmetries turns out to be related, in one way or another, to the tensor square of some graph algebra. For instance  $Oc(E_6)$  is the tensor square of the graph algebra of  $E_6$  taken above the graph subalgebra generated by the ambichiral vertices 0, 4, 3. We remind the reader that  $E_6$  admits self - fusion, with graph algebra given by the following table.



*	<u>0</u>	<u>3</u>	<u>4</u>	<u>1</u>	<u>2</u>	<u>5</u>
<u>0</u>	<u>0</u>	<u>3</u>	<u>4</u>	<u>1</u>	<u>2</u>	<u>5</u>
<u>3</u>	<u>3</u>	<u>0 + 4</u>	<u>3</u>	<u>2</u>	<u>1 + 5</u>	<u>2</u>
<u>4</u>	<u>4</u>	<u>3</u>	<u>0</u>	<u>5</u>	<u>2</u>	<u>1</u>
<u>1</u>	<u>1</u>	<u>2</u>	<u>5</u>	<u>0 + 2</u>	<u>1 + 3 + 5</u>	<u>2 + 4</u>
<u>2</u>	<u>2</u>	<u>1 + 5</u>	<u>2</u>	<u>1 + 3 + 5</u>	<u>0 + 2 + 2 + 4</u>	<u>1 + 3 + 5</u>
<u>5</u>	<u>5</u>	<u>2</u>	<u>1</u>	<u>2 + 4</u>	<u>1 + 3 + 5</u>	<u>0 + 2</u>

It is natural to try to realize  $Oc(F_4)$ , that we obtained by solving the modular splitting equation, directly in terms of some analogous algebraic construction. From the fact that  $F_4$  is an orbifold of  $E_6$ , it is easy to make an educated guess, and, by calculating the corresponding multiplication table, check that it is indeed correct. We claim that  $Oc(F_4) = E_6 \dot{\otimes} E_6$  where  $\dot{\otimes}$  denotes the tensor product taken, not above the complex numbers but above the subalgebra  $J$  generated by vertices 0 and 4 of  $E_6$ . In other words, we identify  $a * u \otimes b$  and  $a \otimes u * b$  as soon as  $u \in J$ .

The twenty generators of  $Oc(F_4)$  are realized as follows.

$$\begin{aligned}
\underline{0} &= \underline{0} \dot{\otimes} \underline{0} = \underline{4} \dot{\otimes} \underline{4} = \underline{0} & \underline{6} &= \underline{0} \dot{\otimes} \underline{1} = \underline{4} \dot{\otimes} \underline{5} = \underline{1}' \\
\underline{1} &= \underline{1} \dot{\otimes} \underline{0} = \underline{5} \dot{\otimes} \underline{4} = \underline{1} & \underline{12} &= \underline{0} \dot{\otimes} \underline{2} = \underline{4} \dot{\otimes} \underline{2} = \underline{2}' \\
\underline{2} &= \underline{2} \dot{\otimes} \underline{0} = \underline{2} \dot{\otimes} \underline{4} = \underline{2} & \underline{16} &= \underline{0} \dot{\otimes} \underline{3} = \underline{4} \dot{\otimes} \underline{3} = \underline{3}' \\
\underline{3} &= \underline{3} \dot{\otimes} \underline{0} = \underline{3} \dot{\otimes} \underline{4} = \underline{3} & \underline{11} &= \underline{0} \dot{\otimes} \underline{5} = \underline{4} \dot{\otimes} \underline{1} = \underline{5}' \\
\underline{4} &= \underline{5} \dot{\otimes} \underline{0} = \underline{1} \dot{\otimes} \underline{4} = \underline{5} & & \\
\underline{5} &= \underline{4} \dot{\otimes} \underline{0} = \underline{0} \dot{\otimes} \underline{4} = \underline{4} & & \\
\underline{7} &= \underline{1} \dot{\otimes} \underline{1} = \underline{5} \dot{\otimes} \underline{5} = \underline{11}' & & \\
\underline{10} &= \underline{5} \dot{\otimes} \underline{1} = \underline{1} \dot{\otimes} \underline{5} = \underline{15}' & & \\
\underline{8} &= \underline{2} \dot{\otimes} \underline{1} = \underline{2} \dot{\otimes} \underline{5} = \underline{21}' & \underline{13} &= \underline{1} \dot{\otimes} \underline{2} = \underline{5} \dot{\otimes} \underline{2} = \underline{12}' \\
\underline{14} &= \underline{2} \dot{\otimes} \underline{2} = \underline{22}' & & \\
\underline{9} &= \underline{3} \dot{\otimes} \underline{1} = \underline{3} \dot{\otimes} \underline{5} = \underline{31}' & \underline{17} &= \underline{1} \dot{\otimes} \underline{3} = \underline{5} \dot{\otimes} \underline{3} = \underline{13}' \\
\underline{15} &= \underline{3} \dot{\otimes} \underline{2} = \underline{32}' & \underline{18} &= \underline{2} \dot{\otimes} \underline{3} = \underline{23}' \\
\underline{19} &= \underline{3} \dot{\otimes} \underline{3} = \underline{33}' & &
\end{aligned}$$

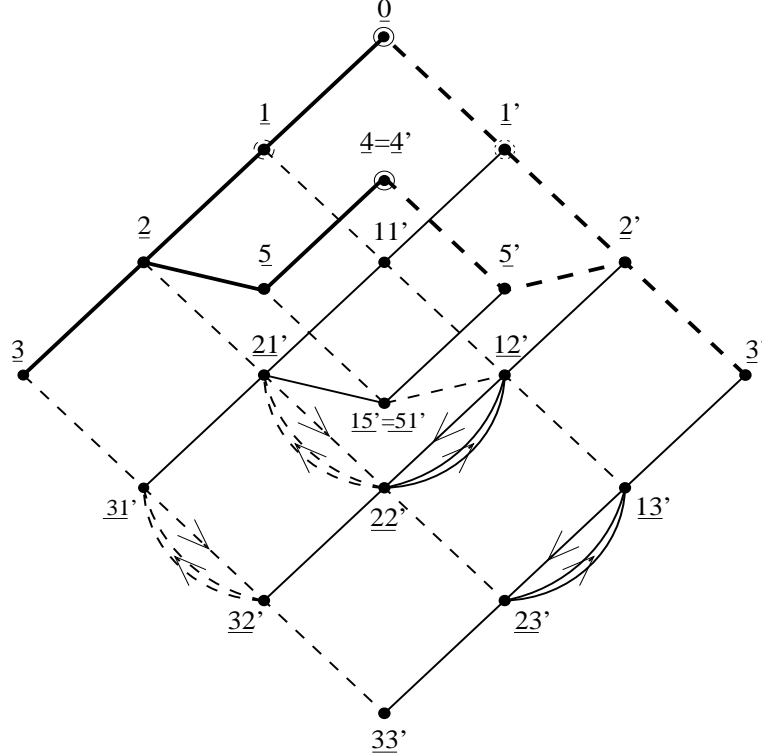
Labels on the left correspond to the original notation  $\{0, 1, 2 \dots 19\}$  that we have been using before, while underlined labels on the right refer to tensor products of  $E_6$  vertices (as defined<sup>7</sup> by the above  $E_6$  diagram, like in [12], [2] or [19]). Because of this labeling convention, notice that  $4 = \bar{5}$  and  $5 = \bar{4}$ . Using the above realization, one recovers the multiplication table of quantum symmetries. For instance,  $\underline{21'} \times \underline{23'} = (\underline{22'})_2 + (\underline{2'})_2$ . Indeed,

$$8 \times 18 = (\underline{2} \otimes \underline{1}) \times (\underline{2} \otimes \underline{3}) = (\underline{2*2}) \otimes (\underline{1*3}) = (\underline{0+2+2+5}) \otimes (\underline{2}) = (\underline{0 \otimes 2})_2 + (\underline{2 \otimes 2})_2 = 12 + 12 + 14 + 14$$

We therefore recover the matrix product equality  $O_8 \times O_{18} = 2O_{12} + 2O_{14}$

## The Ocneanu graph

Using  $E_6 \otimes E_6$  notation for the vertices, the  $F_4$  Ocneanu graph is given as follows:



It is the Cayley graph of multiplication of the linear generators of  $Oc$  by the two generators  $O_1$  and  $O'_1$ , called the *chiral* generators. It is the union of two distinct graphs called left and right graphs, involving the same set of vertices. They are drawn in two different colors (or solid and dashed lines). One can obtain one graph from the other by performing a symmetry with respect to the vertical axis. The six vertices that belong to this axis of symmetry are the *self-dual points*.

The subalgebra generated by the unit and the left chiral generator  $\underline{1}$  is the left chiral subalgebra. It is spanned by vertices  $\{0, \underline{1}, \underline{2}, \underline{5}, \underline{4}, \underline{3}\}$  and corresponds to the connected component of the identity of the left chiral graph. The subalgebra generated by the unit and the right chiral generator  $\underline{1'}$  is the right chiral subalgebra. It is spanned by vertices  $\{0, \underline{1'}, \underline{2'}, \underline{5'}, \underline{4'}, \underline{3'}\}$  and corresponds to the connected component of the identity of the right chiral graph. The intersection of these two

<sup>7</sup>Warning: There are several conventions in the literature

subalgebras is spanned by the set of *ambichiral points*, namely the two-element set  $\{\underline{0}, \underline{4}\}$ . These points are self-dual.

Because of these symmetry considerations, we discuss only the left graph. It is itself given by the union of four disjoint connected graphs, two of type  $E_6$ , that we call respectively  $E = E_6[1]$  (the left chiral subalgebra) and  $e = E_6[2]$  (span of  $\underline{1'}, \underline{11'}, \underline{21'}, \underline{51'}, \underline{5'}, \underline{31'}$ ), and two of type  $F_4$ , that we call  $F = F_4[1]$  (span of  $\underline{32'}, \underline{22'}, \underline{12'}, \underline{2'}$ ) and  $f = F_4[2]$  (span of  $\underline{33'}, \underline{23'}, \underline{13'}, \underline{3'}$ ). The description of the right graph is similar. Notice that  $F_4$  Dynkin diagrams emerge from our resolution of the equations of modular splitting.

The first  $E_6$  (left) subgraph called  $E$  describes the subalgebra generated by  $O_1$ . The other  $E_6$  (left) subgraph called  $e$  is not a subalgebra of  $Oc$  but a module over  $E$ . The two subgraphs of type  $F_4$  called  $F$  and  $f$  are also modules over  $E$ , but their properties are very different. Writing the full multiplication table would be too long, but the interested reader can easily do it, either using  $20 \times 20$  matrices, or, more simply, using the multiplication table of the graph algebra of  $E_6$  together with the realization of generators of  $Oc(F_4)$  in terms of tensor products (see previous section). We have the following relations between the different subspaces:

$\times$	$E$	$e$	$F$	$f$
$E$	$E$	$e$	$F$	$f$
$e$	$e$	$E + F$	$e + f$	$F$
$F$	$F$	$e + f$	$E + F$	$e$
$f$	$f$	$F$	$e$	$E$

## 4 Actions, coactions and sum rules

As written in the introduction, the Ocneanu quantum groupoid associated with a simply laced diagram  $G$  (with  $r$  vertices) belonging to the  $SU(2)$  system, or to an higher system, possesses two – usually distinct – algebras of characters. The first, called the fusion algebra of  $G$ , and denoted  $A(G)$ , can be identified with the graph algebra of the graph  $A_n$  (for a proper choice of  $n$ ). The second algebra of characters, called the algebra of quantum symmetries, is denoted  $Oc(G)$ . The fusion algebra  $A(G) = A_n$  acts on the vector space spanned by the vertices of the graph  $G$ , and this action is explicitly described by the so-called “fused adjacency matrices”  $F_p$ . These matrices have therefore the same commutation relations as the fusion matrices  $N_p$  (the generators of  $A_n$ ) but their size is smaller since it is  $r \times r$ , where  $r$  is the number of vertices of  $G$ . In the same way, but at the dual level (coaction), the algebra of quantum symmetries  $Oc(G)$  acts on  $G$  and this is described by the so-called “quantum symmetry fused matrices”  $\Sigma_x$  of  $G$ . These matrices have therefore the same commutation relations as the quantum symmetry matrices  $O_x$  (the generators of  $Oc(G)$ ) but their size is smaller since it is again  $r \times r$ . See [2], [3] for explicit expressions of these matrices in various  $ADE$  cases. In the simply laced situation, there is a general theory [12] (see also [6], [18]) that tells us how to build first, a product law – composition – defined as composition of graded endomorphisms of essential paths (“horizontal paths”) on the given graph, then a co-product law – associated, via the choice of a scalar product, to convolution – by using the composition of endomorphisms of the so - called “vertical paths”. However, to our knowledge, for a non simply-laced diagram like the one we study here, the general theory is not known, yet. Our purpose, in this section, is therefore very modest, in the sense that we shall only mimic what we would have done in the simply laced situation, and describe what we find. This is admittedly rather naive, since when counting dimensions, for instance, we take the oriented double line of the  $F_4$  diagram (between vertices  $a_2$  and  $a_1$ ) as a pair of two essential paths of length one, and this is maybe not what should be done.

## Fused matrices $F_p$ relative to the fusion generators $N_p$ of $A_{11}$

We first consider the action of  $A_{11}$  implemented by matrices  $F_p$ , to be found. The simplest determination stems from the fact that  $A_{11}$  is a truncated version of the algebra of characters of  $SU(2)$ , so that the  $F_p$ 's are obtained from the usual recurrence formula (composition of spins)  $F_p F_1 = F_{p-1} + F_{p+1}$ , and the seed:  $F_0 = \mathbb{1}_4$  and  $F_1$  is equal to the adjacency matrix  $G_1$  of the graph  $F_4$ . This recurrence relation has a period  $(2 \times 12)$  and one can check that  $F_{10} F_1 = F_9$  since  $F_{11}$  vanishes. For  $12 \leq p \leq 23$ ,  $F_{p+12} = -F_p$  but we are only interested here in the positive part. Notice that  $F_1$  is not symmetric, since the graph  $F_4$  itself, with vertices  $a_0, a_1, a_2, a_3$ , is not<sup>8</sup> (two oriented edges between vertices  $a_2$  and  $a_1$  but only one oriented edge between  $a_1$  and  $a_2$ ). We obtain the following 11 matrices and check that  $(F_p F_q) a_i = F_p (F_q a_i)$ , as it should.

$$\begin{array}{cccccc}
 F_0 & & F_1 & & F_2 & & F_3 & & F_4 & & F_5 \\
 \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} & & \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} & & \begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & 2 & \cdot & 2 \\ 1 & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} & & \begin{pmatrix} \cdot & 1 & \cdot & 2 \\ 1 & \cdot & 4 & \cdot \\ \cdot & 2 & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix} & & \begin{pmatrix} 1 & \cdot & 2 & \cdot \\ \cdot & 3 & \cdot & 2 \\ 1 & \cdot & 3 & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix} & & \begin{pmatrix} \cdot & 2 & \cdot & \cdot \\ 2 & \cdot & 4 & \cdot \\ \cdot & 2 & \cdot & 2 \\ \cdot & \cdot & 2 & \cdot \end{pmatrix}
 \end{array}$$

and  $F_p = F_{10-p}$  for  $p = 6, \dots, 10$ . From the fused adjacency matrices  $F_p$  we can obtain four essential matrices  $E_a$  that are rectangular  $4 \times 11$  matrices defined by  $(E_a)_{b,n} = (F_n)_{a,b}$ . The  $F$ 's and the  $E$ 's determine the induction/restriction rules between the graphs  $A_{11}$  and  $F_4$ .

## Fused matrices $\Sigma_x$ relative to the quantum symmetry generators $O_x$ of $O_c$

We now turn to the determination of matrices  $\Sigma_x$ . We shall present two methods. The most direct uses the fact that these matrices provide a  $4 \times 4$  realization of  $O_c$ . It is enough to define  $\Sigma_0 = \mathbb{1}_4$ , to set  $\Sigma_1 = \Sigma_6$  equal to the adjacency matrix of the  $F_4$  diagram and use the same relations that determine all matrices  $O_x$  from  $O_0$ ,  $O_1$  and  $O_6$  (equivalently, solve the system of equations  $\Sigma_x \Sigma_y = O_{xyz} \Sigma_z$  with given structure constants  $O_{xyz}$ ).

Another method uses our realization of  $O_c(F_4)$  as a fibered tensor product  $E_6 \dot{\otimes} E_6$ . Remember that the (left, for instance) graph of quantum symmetries is a union of four graphs  $E, e, F, f$ , two of type  $E_6$ , two of type  $F_4$ , so that any single connected component describes a module action over the subalgebra associated with the first subgraph ( $E$ ). In this way we obtain four sets of matrices:  $s_u^E$  (of dimension  $6 \times 6$ ),  $s_u^e$  (of dimension  $6 \times 6$ ),  $s_u^F$  (of dimension  $4 \times 4$ ), and  $s_u^f$  (of dimension  $4 \times 4$ ). In all cases,  $u$  runs from 0 to 5, so that these are six-elements sets. The elements of the first set  $s^E$  coincide with the already known generators of the graph algebra of  $E_6$ . What we have to use in this section is the second module<sup>9</sup> of type  $F_4$  (called  $f$ ) and the matrices  $s^f$  that express the action  $E \times f \subset f$ . Remember also that we have  $f \times f \subset E$ . These matrices are as follows:

$$s_0^f = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad s_1^f = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \quad s_2^f = \begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & 2 & \cdot & 2 \\ 1 & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \quad s_3^f = \begin{pmatrix} \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

and  $s_4^f = s_1^f$ ,  $s_5^f = s_0^f$ . From the expressions giving the linear generators of  $O_c$  as tensor products of  $E_6$  vertices, we obtain  $\Sigma_x = s_a \cdot s_b$ , whenever  $x = a \dot{\otimes} b$ .

<sup>8</sup>The reader may check that  $C = 2\mathbf{1} - G_1$  is the usual Cartan matrix of  $F_4$

<sup>9</sup>In this respect the situation is similar to the analysis of  $E_7$ , which appears as a subgraph of its own algebra of quantum symmetries and as a module over the graph algebra of  $D_{10}$ , itself a subalgebra of  $O_c(E_7)$

The two methods give the same result and we obtain the following 20 matrices:

$$\begin{array}{cccccc} \Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_3 & \Sigma_7 & \Sigma_8 \\ \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} & \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} & \begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & 2 & \cdot & 2 \\ 1 & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} & \begin{pmatrix} \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & 2 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} & \begin{pmatrix} 1 & \cdot & 2 & \cdot \\ \cdot & 3 & \cdot & 2 \\ 1 & \cdot & 3 & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix} & \begin{pmatrix} \cdot & 2 & \cdot & 2 \\ 2 & \cdot & 6 & \cdot \\ \cdot & 3 & \cdot & 2 \\ 1 & \cdot & 2 & \cdot \end{pmatrix} \end{array}$$

and  $\Sigma_5 = \Sigma_{19}/2 = \Sigma_0$ ,  $\Sigma_4 = \Sigma_6 = \Sigma_{11} = \Sigma_{15}/2 = \Sigma_{18}/2 = \Sigma_1$ ,  $\Sigma_9 = \Sigma_{12} = \Sigma_{17} = \Sigma_2$ ,  $\Sigma_{16} = \Sigma_3$ ,  $\Sigma_{10} = \Sigma_{14}/2 = \Sigma_7$ ,  $\Sigma_{13} = \Sigma_8$ . Notice that  $\Sigma_{14}, \Sigma_{15}, \Sigma_{18}, \Sigma_{19}$ , as matrices over positive integers, can be divided by 2.

As another check of the correctness of the previous calculation, there exists a relation that holds between matrices  $F_p$  and matrices  $\Sigma_x$ . It could actually be used to determine the former from the later, although this method would be more complicated than the one we followed. This relation reads:  $F_n = \sum_y (W_{\underline{0},y})_{n,0} \Sigma_y$  and stems from the compatibility between actions of  $A_{11}$  and  $Oc$  on the diagram  $F_4$  and the fact that the unit  $\underline{0}$  of  $Oc$  indeed acts trivially.

## Sum rules

- Quadratic sum rule. Being both semi-simple and co-semi-simple, the following quadratic sum rule holds for the Ocneanu quantum groupoid associated with a simply laced diagram:  $\sum_p d_p^2 = \sum_x d_x^2$ , where  $d_p = \sum_{a,b} (F_p)_{a,b}$  gives the dimensions of the simple blocks for the first algebra structure, and  $d_x = \sum_{a,b} (\Sigma_x)_{a,b}$  gives the dimensions of the simple blocks for the second algebra structure (actually the algebra structure defined on the dual). If we take  $G = E_6$  for instance, we get  $d_p = \{6, 10, 14, 18, 20, 20, 20, 18, 14, 10, 6\}$ ,  $d_x = \{6, 8, 6, 10, 14, 10, 10, 14, 10, 20, 28, 20\}$  and we check that  $\sum_p d_p^2 = \sum_x d_x^2 = 2512$ .

In the case of the non simply laced diagram  $G = F_4$ , analogous calculations for  $d_p$  and  $d_x$  lead to the following values:

$$\begin{aligned} d_p &= \{4, 7, 10, 13, 14, 14, 14, 13, 10, 7, 4\} \\ d_x &= \{4, 7, 10, 6, 7, 4, 7, 14, 20, 10, 14, 7, 10, 20, 28, 14, 6, 10, 14, 8\} \end{aligned}$$

Notice that  $\sum_p d_p^2 = 1256$ , which is half the  $E_6$  result; this could be expected since  $F_4$  is a  $\mathbb{Z}_2$  orbifold of  $E_6$ . However, this value is not equal to  $\sum_x d_x^2$ . This could also be expected if we remember that vertices 14, 15, 18, 19 play a special role (like the “long roots” in the theory of Lie algebras): these are the values for which  $W_{0,x} = 2W_{x,0}$  (no factor 2 for the others) and for which the  $\Sigma_x$  matrices can be divided by 2. For this reason, we introduce another set of matrices, setting  $\tilde{\Sigma}_x = \Sigma_x/2$  for those four values, and equality otherwise. We also introduce the corresponding dimensions  $\tilde{d}_x$ , which are equal to  $d_x$  except for the four special vertices where the values are divided by 2. We find  $\sum_x d_x \tilde{d}_x = 2512$  and notice that this value is twice the value of the sum  $\sum_p d_p^2$ . It would be natural to introduce a quadratic form in the vector space  $Oc$ , diagonal and taking the value 1 on the basis generators  $O_x$ , except in positions 15, 16, 19, 20 where the coefficients would be equal to 2. The conclusion is that the usual quadratic sum rule almost works, in the sense that it is somehow twisted by the appearance of factors 2 which should be understood from the fact that basis generators  $O_x$  of quantum symmetries have two different lengths (corresponding to short and long roots in the theory of Lie algebras). These results are not compatible with the existence of a quantum groupoid structure (in the usual sense) since the candidates for algebras of characters associated with the two multiplicative structure – that would be respectively described by the semi-simple algebras  $Oc$  (20 blocks) or the direct sum of two copies of the graph algebra  $A_{11}$  (twice 11 blocks) – do not have the same dimension. A more general type of algebraic structure seems to be needed.



- Linear sum rule. It is an observational fact (not yet understood) that the following linear sum rule also holds<sup>10</sup>:  $\sum_p d_p = \sum_x d_x$ , for most *ADE* cases; and when it does not, one also knows how to “correct” the rule by introducing natural prefactors. In the case of  $E_6$ , for instance, this sum equals 156. For the graph  $F_4$  however,  $\sum_p d_p = 110$  whereas  $\sum_x d_x = 220$ . This is also compatible with the previous discussion.
- Quantum sum rule. For *ADE* diagrams  $G$  with  $n$  vertices  $\sigma_i$  the quantum mass  $m(G)$  is defined by:

$$m(G) = \sum_{a=0}^{n-1} (qdim(\sigma_i))^2$$

where the quantum dimensions  $qdim$  of the vertex  $\sigma_i$  is given by the  $i$ -component of the normalized Perron-Frobenius vector, associated with the highest eigenvalue (here  $\beta = \frac{1+\sqrt{3}}{\sqrt{2}}$ ). To get these quantities for the vertices of  $Oc$ , we assign  $\beta$  to both chiral generators, impose that  $qdim$  is an algebra morphism and use recurrence formulae for  $O_x$ .

For *ADE* cases, the following property can be verified (see [19]) : if we denote the fusion algebra of the graph  $G$  by  $A(G)$  (a graph algebra of type  $A_n$ ) and the algebra of quantum symmetries by  $Oc(G)$ , one finds that  $m(A(G)) = m(Oc(G))$ . Moreover, for a graph with self-fusion, and if it is so that  $Oc(G)$  is isomorphic, as an algebra, with  $G \otimes_J G$ , then  $m(Oc(G)) = (m(G) \times m(G))/m(J)$ . For instance in the  $E_6$  case

$$m(E_6) = 4(3 + \sqrt{3}) \quad \text{and} \quad m(Oc(E_6)) = \frac{m(E_6) \times m(E_6)}{m(J)} = 24(2 + \sqrt{3}) = m(A_{11})$$

However, for the non simply laced diagram  $F_4$ , the  $qdim$  are as follows ( $x = 0, \dots, 19$ )

$$qdim\{E, e, F, f\} = \left\{ \left(1, \frac{1+\sqrt{3}}{\sqrt{2}}, 1 + \sqrt{3}, \sqrt{2}, \frac{1+\sqrt{3}}{\sqrt{2}}, 1\right), \left(\frac{1+\sqrt{3}}{\sqrt{2}}, 2 + \sqrt{3}, \sqrt{2}(2 + \sqrt{3}), 1 + \sqrt{3}, 2 + \sqrt{3}, \frac{1+\sqrt{3}}{\sqrt{2}}\right), \right. \\ \left. (1 + \sqrt{3}, \sqrt{2}(2 + \sqrt{3}), 2(2 + \sqrt{3}), \sqrt{2}(1 + \sqrt{3})), (\sqrt{2}, 1 + \sqrt{3}, \sqrt{2}(1 + \sqrt{3}), 2) \right\}$$

Like for the quadratic sum rule we introduce quantum dimensions  $\widetilde{qdim}(x)$  equal to  $qdim(x)$  except for the four vertices 14, 15, 18, 19 where the values are divided by 2. One finds<sup>11</sup>:

$$m(Oc(F_4)) = \sum_{a=0}^{n-1} \left( qdim(x) \widetilde{qdim}(x) \right)^2 = m(E) + m(e) + m(F) + m(f) = 48(2 + \sqrt{3}) = 2m(A_{11})$$

with  $m(E) = m(f) = 4(3 + \sqrt{3})$  and  $m(e) = m(F) = 4(9 + 5\sqrt{3})$ .

- Quadratic modular double sum rule. The modular splitting relation implies the following. Call  $d_p^N = \sum_{q,r} (N_p)_{q,r}$ ,  $d_x^{W'} = \sum_{y,z} (W_{x,0})_{y,z}$  and  $d_x^{W''} = \sum_{y,z} (W_{0,x})_{y,z}$ , then (take traces) :

$$\sum_{p,q} d_p^N d_q^N M_{p,q} = \sum_x d_x^{W'} d_x^{W''}$$

Once the  $W_x$  are determined, one should verify that this sum rule holds. In the simply laced case  $E_6$ , for instance, one easily checks this identity, with  $d_x^{W'} = d_x^{W''}$  for all  $x$  given by  $d^W = \{20, 28, 20, 20, 28, 20, 12, 16, 12, 34, 48, 34\}$  (sum of squares is 8328) and  $d^N = \{11, 20, 27, 32, 35, 36, 35, 32, 27, 20, 11\}$  for  $A_{11}$  so that the  $M$ -norm square defined by the  $E_6$  modular matrix  $M$  is also 8328. In the case of  $F_4$  we have the same dimension vector  $d^N$  with its  $M$ -norm square (equal to 4232) now defined by the  $M$  matrix of  $F_4$ , but  $d_x^{W'} \neq d_x^{W''}$  and we have to take  $d^{W'} = \{8, 12, 16, 8, 12, 8, 12, 18, 24, 12, 18, 12, 16, 24, 16, 8, 8, 12, 8, 4\}$  together with  $d^{W''} = \{8, 12, 16, 8, 12, 8, 12, 18, 24, 12, 18, 12, 16, 24, 32, 16, 8, 12, 16, 8\}$  – notice the factor 2 for entries 15, 16, 19, 20 – so that the right hand side of this sum rule is also 4232, as it should.

<sup>10</sup>This was noticed in [18]

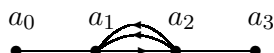
<sup>11</sup>Using the graph algebra of  $F_4$  defined in the following section, one finds rather  $m(F_4) = m(E_6)/2$

## 5 Miscellaneous comments

### Comparison with a direct method using self-fusion on $F_4$

Remember that the diagram  $F_4$  emerged from our analysis of the modular splitting equation and that we started from a given partition function. Now, we would like to reverse the machine and start from the diagram  $F_4$  itself. We imitate techniques initiated in [2] and developed in [3], [19].

- From the diagram, we find its adjacency matrix and call it  $G_1$ . From its eigenvalues  $2\cos(r\pi/12)$ , we find the exponents  $r = 1, 5, 7, 11$ . In particular the highest eigenvalue is  $\beta = \frac{1+\sqrt{3}}{\sqrt{2}} = 2\cos(\frac{\pi}{\kappa})$  with  $\kappa = 12$  gives the value of the Coxeter number – not the dual Coxeter number, which is different (and equals 9) since the diagram is not simply laced. The quantum dimensions of the vertices are given by the normalized Perron-Frobenius vector associated with  $\beta$ , and we obtain the  $q$ -numbers  $[1], [2], [2], [1]$ , for  $q = e^{\frac{i\pi}{\kappa}}$ , so  $q^{2 \times 12} = 1$ .
- The fused adjacency matrices  $F_p$  are obtained by checking that the vector space of the diagram  $F_4$  is indeed an  $A_{11}$  module and by imposing the usual  $SU(2)$  recurrence relation for the  $F_p$ 's, together with the seed  $F_0 = \mathbb{1}_4$  and  $F_1 = G_1$ . Vertices of the diagram  $F_4$  are labelled as follows:



a.) Oriented graph  $F_4$ .



b.) Coxeter – Dynkin  $F_4$  diagram.

- One is tempted to analyze the possibility of defining a graph algebra structure for the diagram  $F_4$ . This is indeed possible. The multiplication table given below is determined by imposing associativity, once the multiplications by 0 (unity) and 1 (adjacency matrix) have been defined.

$\cdot$	$a_0$	$a_1$	$a_2$	$a_3$
$a_0$	$a_0$	$a_1$	$a_2$	$a_3$
$a_1$	$a_1$	$a_0 + a_2$	$2a_1 + a_3$	$a_2$
$a_2$	$a_2$	$2a_1 + a_3$	$2a_2 + 2a_0$	$2a_1$
$a_3$	$a_3$	$a_2$	$2a_1$	$2a_0$

The graph matrices  $G_a$ ,  $a = 0, 1, 2, 3$ , obtained from this table, coincide with the first four matrices  $\Sigma_x$ , but the reader will notice immediately that this table cannot be obtained, by restriction, from the multiplication table of  $Oc$ . Indeed, there are only two candidates ( $F$  or  $f$ ). The second one is ruled out by the fact that  $f \times f \subset E$ . The first one is not a subalgebra either since  $F \times F \subset E + F$ , and even if we artificially project the result (right hand side) to  $F$ , the obtained table will differ from the one just given. The conclusion is that there is no hope to use the above graph algebra structure on  $F_4$  to recover the modular matrix  $M$  that we used as the starting point of the whole analysis carried out in this paper. Let us nevertheless proceed.

- Potential candidates for the ambichiral vertices can be obtained by imposing that the eigenvalues of the  $T$  modular operator (they are well defined for the vertices of  $A_{11}$ ) are also well

defined under the induction rules (see [4] for details and examples). This constraint selects the set  $\{a_0, a_3\}$  so that a natural guess for the corresponding algebra of quantum symmetries would be  $F_4 \otimes F_4$  where the algebra structure of each factor was described in the previous paragraph and where the tensor product is taken above the subalgebra generated by  $\{a_0, a_3\}$ . The new modular matrix  $M^{new} = (W_{0,0})^{new}$  is then given by  $E_0^{red}(E_0^{red})^T$  where the reduced essential matrix  $E_0^{red}$  is obtained from  $E_0$  by keeping only the first and last column and replacing the two others by zeros. It is equal to

$$M^{new} = \begin{pmatrix} 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \end{pmatrix}$$

As expected, it differs from the matrix  $M = W_{\underline{0}} = W_{0,0}$ . However it is interesting to notice that both are related by a conjugacy :  $M^{new}/2 = S^{-1} M S$ , where  $S$  stands for one of the two generators of the modular group in this representation. We find that the bilinear form obtained from  $W_{00}$  is invariant under the action of the congruence subgroup  $S^{-1}\Gamma_0^{(2)}S = \text{gen}\{S^{-1}T^2S, S^{-1}(ST^{-2}S)S\}$  conjugated with  $\Gamma_0^{(2)}$ . This can be directly verified by calculating the commutators

$$[S^{-1}T_{11}^2S, W_{00}] = 0 \quad , \quad [S^{-1}(ST^{-2}S)_{11}S, W_{00}] = 0$$

Notice that  $M^{new}$  (which has 20 non-zero entries) is equal to the sum of the three matrices  $W_{\underline{0}}$ ,  $W_{\underline{4}}$  and  $W_{\underline{33}}$  associated with three self-dual points of the graph  $Oc(F_4)$ .

So, we can indeed define self-fusion on the diagram  $F_4$  but this associative algebra structure does not seem to be simply related with the so-called  $F_4$  modular matrix. Still another possibility would be to work with a symmetrized form of the  $F_4$  diagram, i.e., with an “adjacency matrix” that incorporates non-integer matrix elements ( $q$ -numbers equal to  $\sqrt{2}$ ). This possibility is actually quite interesting but will not be discussed here. It does not seem to allow one to recover the  $F_4$  modular matrix  $M$ , either.

## A relative equation of modular splitting

From the fact that the diagram  $F_4$  is, geometrically, a  $Z_2$  orbifold of  $E_6$  (identify pair of vertices  $(0, 4)$ ,  $(1, 5)$  of the later), we are tempted to consider an action of the graph algebra of  $E_6$  (this graph has indeed self-fusion) on the vector space of  $F_4$ . Call  $G_u^{E_6}$  the six  $6 \times 6$  generators of the  $E_6$  graph algebra and  $F_u^{E_6}$  the six  $4 \times 4$  matrices implementing the action of  $E_6$  on  $F_4$ . The multiplication table of  $E_6$  was given before. Its graph matrices obey the usual relations:

$$G_0^{E_6} = \mathbb{1}_6, \quad G_1^{E_6}, \quad G_2^{E_6} = G_1^{E_6} \cdot G_1^{E_6} - G_0^{E_6}, \\ G_4^{E_6} = G_1^{E_6} \cdot G_1^{E_6} \cdot G_1^{E_6} \cdot G_1^{E_6} - 4G_1^{E_6} \cdot G_1^{E_6} + 2G_0^{E_6}, \quad G_5^{E_6} = G_1^{E_6} \cdot G_4^{E_6}, \quad G_3^{E_6} = -G_1^{E_6} \cdot (G_4^{E_6} - G_1^{E_6} \cdot G_1^{E_6} + 2G_0^{E_6})$$

To obtain the fused matrices  $F_u^{E_6}$  relative to this action, we set  $F_0^{E_6} = \mathbb{1}_4$ ,  $F_1^{E_6} = G_1$  (the adjacency matrix of  $F_4$ ) and impose that the  $F_u^{E_6}$  should obey the same algebra relations as the  $G_u^{E_6}$ .

Exactly as we had an action of  $A_{11}$  on  $F_4$ , implemented by matrices  $F_p$ , we have a (relative) action of  $E_6$  on  $F_4$ , implemented by matrices  $F_u^{E_6}$ . For this reason we are led to consider a “relative”

theory of modular splitting (and a corresponding equation) with  $A_{11}$  replaced by  $E_6$ . In particular the graph matrices  $N_p$  of  $A_{11}$  – the usual fusion matrices – are replaced by the generators  $G_u^{E_6}$  of the graph algebra of  $E_6$ . With  $u, v \in E_6$ , we define “relative” toric matrices by

$$u x v = \sum_y (W_{x,y}^{E_6})_{u,v} y$$

The relative equation of modular splitting reads ( $\tau$  is a tensorial flip):

$$\sum_{u,v} G_u^{E_6} \otimes G_v^{E_6} M_{u,v}^{rel} = \sum_{x \in O_c} \tau \circ (W_{x,0}^{E_6} \otimes W_{0,x}^{E_6})$$

$M^{rel}$  describes the same  $F_4$  partition function as before, but in terms of generalized characters<sup>12</sup>:

$$\chi_0 + \chi_6, \quad \chi_1 + \chi_5 + \chi_7, \quad \chi_2 + \chi_4 + \chi_6 + \chi_8, \quad \chi_3 + \chi_5 + \chi_9, \quad \chi_4 + \chi_{10}, \quad \chi_3 + \chi_7$$

If  $P$  denotes the matrix of this linear transformation (it is the first essential matrix, i.e., the “intertwiner” of the  $E_6$  theory), we have  $M = PM^{rel}P^T$ . The  $E_6$  invariant, in terms of these generalized characters, with the above ordering, is diagonal and reads  $diag(1, 0, 0, 0, 1, 1)$  whereas the  $F_4$  modular matrix is  $M^{rel} = diag(1, 0, 0, 0, 1, 0)$ . The equation of modular splitting is then solved exactly as we did in a previous section, with the technical advantage that the size of the matrices that we have to manipulate is much smaller ( $36 \times 36$  rather than  $121 \times 121$ ). Same comment for most objects of the theory: the analogue of  $K_0$  is  $36 \times 20$  (rather than  $121 \times 20$ ) and the relative toric matrices are  $6 \times 6$  (rather than  $11 \times 11$ ). The twenty generators  $O_x$  are the same (their size is  $20 \times 20$ ) and the graph of quantum symmetries is determined as before. It is easy to translate the relative  $E_6$  theory to the “usual” one (that uses the action of the  $A_{11}$  graph algebra) by using the – rectangular – essential matrix  $P$ . It is technically easier to work with this relative theory, but the drawback is that the  $E_6$  case should be analyzed first. This is why we did not follow this method in our presentation.

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<sup>12</sup>In our formalism, they are obtained from essential matrices of the  $E_6$  diagram as described, for instance, in [5]

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